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Scale covariant physics: a 'quantum deformation' of classical electrodynamics

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Abstract

We present a deformation of classical electrodynamics, continuously depending on a 'quantum parameter', featuring manifest gauge, Poincaré and scale covariance. The theory, dubbed extended charge dynamics (ECD), associates a certain length scale with each charge which, due to scale covariance, is an attribute of a solution, not a parameter of the theory. When the EM field experienced by an ECD charge is slowly varying over that length scale, the dynamics of the charge reduces to classical dynamics, its emitted radiation reduces to the familiar Liénard–Wiechert potential and the above length scale is identified as the charge's Compton length. It is conjectured that quantum mechanics describes statistical aspects of ensembles of ECD solutions, much like classical thermodynamics describes statistical aspects of ensembles of classical solutions. A unique 'remote sensing' feature of ECD, supporting that conjecture, is presented, along with an explanation for the illusion of a photon within a classical treatment of the EM field. Finally, a novel conservation law associated with the scale covariance of ECD is derived, indicating that the scale of a solution may 'drift' with time at a constant rate, much like translation covariance implies a uniform drift of the (average) position.

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1. Introduction

1.1. The Klein–Nishina–Thompson coincidence

When electromagnetic (EM) radiation of wavelength λ_R impinges on an electron of mass m, the differential cross section for scattering of EM energy into a given solid angle is given by the celebrated Klein–Nishina (KN) formula. That formula, derived from quantum electrodynamics (QED), involves a second length scale—the electron's Compton length,

$\lambda_e = \hbar/(mc)$. For $\lambda_e \ll \lambda_R$, the KN formula reduces to the classical, namely, non-quantum, Thompson formula.

Reduction of a general expression obtained within one theory, to another expression obtained in an approximate theory, is rather common in physics. For example, relativistic mechanics reduce to Newtonian mechanics in the limit of small velocities (compared to the speed of light). However, the reduction of the Klein–Nishina formula to the Thompson formula (along with many other classical $\leftrightarrow$ quantum coincidences) is more subtle. Relativistic mechanics can be 'deformed' (by taking $c \to \infty$) into Newtonian mechanics in a continuous way. In both theories the mathematical objects are spacetime trajectories. In contrast, the derivation of the Thompson formula only assumes a point-like electron (classically) radiating as a result of oscillations in the incident EM wave, whereas the KN formula necessitates the full QED apparatus, along with its distinct language—photons, etc—having no such trajectories in its vocabulary (even in the limit $\hbar \to 0$). The remarkable thing is that both calculations lead to the same result for $\lambda_e \ll \lambda_R$ despite the use of two manifestly distinct languages.

The above coincidence invites the following conjecture. The electron is not point-like, but rather has a non-vanishing associated length-scale on the order of λ_e. The dynamics of such an 'extended charge' reduces to classical dynamics of a point charge when the EM field experienced by the charge is slowly varying on the Compton length scale. Quantum mechanical calculations, such as the KN formula, describe statistical aspects of ensembles of such extended-charge solutions. In particular, QM must correctly describe those statistical aspects when the dynamics is well approximated by classical electrodynamics—hence the above KN–Thompson coincidence.

The conjectured dynamics of such extended charges, hitherto dubbed extended charge dynamics (ECD), must obviously pass other tests, beyond reduction to classical electrodynamics in the appropriate limit, in order to qualify as a 'hidden variables' model for quantum mechanics. In particular, the violent way by which a charge is jolted in Compton scattering, by even the most minute radiation field, must be explained without resorting to a corpuscular description of light. Indeed, we show that, for $\lambda_e \gg \lambda_R$, the dynamics of an ECD charge may become highly non-classical, and even violent, even in the most feeble external field. In this regime, the strength of the EM field only determines the likelihood of a violent scattering event, not its energy and momentum (or distribution thereof). Of course, to salvage energy–momentum conservation, novel, non-classical conservation laws must exist. These are derived in section 3.4.

Furthermore, ECD is interesting from another perspective: the self-force problem. The singular nature of the current associated with a point charge makes classical electrodynamics an ill-defined theory. Many extensions and deformations of classical electrodynamics aimed at resolving the self-force problem have been proposed over the years, all suffering from one obvious flaw: they contradict experimental results, especially when applied to small scales, where quantum mechanics takes over. As we shall see, due to its unique structure, ECD is defined as the limit of a certain family of finite theories. A mechanism for 'absorbing infinities' in a gauge and Lorentz covariant way is an integral part of the theory, not an external supplement, and the result appears to be consistent with the predictions of quantum mechanics.

1.2. Scale covariant physics

What is a scale covariant physical theory? A central concept to any physical theory is the notion of a spacetime grid, i.e. a hypothetical spatial grid with translation symmetry, carrying a clock at each grid point. The (real valued) labels on the grid and on the dials of the clocks are chosen so as to have the speed of light equal to 1 length-label/time-label, or any other constant also entering the Lorentz transformation. A fairly general definition of a physical theory is a set Σ of permissible functions—abstractions of reality—defined on the spacetime grid, having their value in some field. A physical theory enjoying a symmetry group G is a set Σ which is closed under the group action of G on Σ, induced by relabeling of the grid by any $g \in G$, sending $f(x) \mapsto f(g^{-1}x)$. More generally, if Σ comprises functions having their values in some space V, the action of G on Σ is given by $gf(x) = \rho(g, f(g^{-1}x))$, with $\rho : G \times V \mapsto V$ some group action of G on V, namely, $\rho(g', \rho(g, v)) = \rho(g'g, v) \quad \forall g', g \in G$ and $v \in V$, and the closure of Σ under the action of G is referred to as G-covariance. A real and scalar (having $V = \mathbb{R}$) scale covariant theory, for example, is closed under the dilatation $f(x) \mapsto \lambda^d f(\lambda^{-1}x)$ for any $\lambda > 0$ and some $d \in \mathbb{R}$. (Note that this operation preserves the speed of light.) The scaling transformation is therefore just the constructive multiplication of the labels on the grid by some positive constant, accompanied by a relabeling of V preserving the group property of the dilatation group $D \sim \mathbb{R}^*$. This should be contrasted with a different operation suggested by Nottale [3] in the context of scale covariance, involving also a resolution cutoff.

A theory may be *extended* into a scale covariant one. Denoting by $\Sigma(p)$ the theory defined by parameters $p \in P$ (for example, $\Sigma(p)$ may stand for the solution set of a differential equation with coefficients arranged in a vector $p \in \mathbb{R}^n$) and assuming that for any $\lambda \in D, \lambda\Sigma(p) = \Sigma({}^{\lambda}p)$ with ${}^{\lambda}p \in P$, then the union $\bigcup_{p\in P} \Sigma(p)$ is by construction a theory which is D-covariant. An example we shall briefly analyze is classical electrodynamics parameterized by the charges, q_i, and masses, m_i, of n point charges, which transform under scaling according to $p \equiv (q, m) \mapsto {}^{\lambda}p \equiv (q, \lambda^{-1}m)$. By allowing a charge to have any mass, then, classical electrodynamics can be considered scale covariant.

The fact that electrons come in one particular mass can therefore be interpreted as either implying that physics is not scale covariant or, by the above scale covariant extension, as a 'spontaneously broken' scale covariant theory—the privileged scale (mass) being an attribute of a particular solution, not that of the theory—in the same sense that any localized solution of a translation covariant theory introduces a privileged position. We adopt the latter view. The absence of a privileged scale in a physical theory, we believe, is just as natural a premise as the absence of a privileged position. We ask: why should there be an absolute size, if there is no absolute position (or orientation or velocity)? Is it possible that the introduction of a privileged scale into the laws of physics repeats a mistake made not so long ago with respect to translation covariance? (The Earth—a particular solution of the equations—introduces such a privileged position for anyone conducting experiments on Earth that it was believed to reside at a privileged position—the center of the universe.)

Nevertheless, the scale covariance of the above scale covariant extension of classical electrodynamics is somewhat 'artificial' compared with its manifest translation (more generally Poincaré) covariance. This manifests itself in the absence of an obvious conservation law associated with scale covariance, as opposed to energy–momentum conservation—the significance of which cannot be exaggerated. A central goal of this paper, therefore, is to elevate the status of scale covariance to that of Poincaré covariance. Consequently, ECD features an additional, nontrivial, conservation law associated with scale covariance, on equal footing with energy–momentum conservation. Some speculations pertaining to a possible observational signature of that conservation law are discussed at the end.

2. Classical electrodynamics as an example of a manifestly scale covariant theory

As a first step in the construction of a manifestly scale covariant deformation of classical electrodynamics—ECD—we first show that classical electrodynamics itself can be given a manifestly scale covariant formulation, the (conserved) mass of a charge emerging as an attribute of a specific solution and not as a parameter of the theory. This formulation invites at once a novel conservation law associated with scale covariance.

Consider, then, the following action of n interacting charges via the the EM potential:

$$I = \int \mathrm{d}^4 x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{k=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}s \left[\frac{1}{2} {}^k\dot{\gamma}_s^2 + q^k \dot{\gamma}_s \cdot A({}^k\gamma_s) \right]. \tag{1}$$

Here, $A^\mu : \mathbb{R}^4 \mapsto \mathbb{R}$ ($\mu = 0 \ldots 3$) are the components of the EM vector potential, ${}^k\gamma_s : \mathbb{R} \mapsto \mathbb{R}^4$ ($k = 1 \ldots n$) are n trajectories parametrized by the Lorentz scalar s, q is a coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric Faraday tensor and $u \cdot v$ stands for $g_{\mu\nu} u^\mu v^\nu$ with the metric $g := \mathrm{diag}(1, -1, -1, -1)$. The action I is, in fact, infinite due to the infinite extent of the s integration, but if we restrict a variation of a trajectory ${}^k\gamma$ to vanish for $|s| \to \infty$, then it is still meaningful to seek a trajectory around which the first variation of I vanishes. The action principle associated with (1) is closely related to the so called 'manifestly covariant' formulation of classical electrodynamics (see e.g. [1]), with a small but important difference: no mass-parameter enters the action, and s is *not* the proper time, but rather some arbitrary Lorentz scalar.

The standard Euler–Lagrange prescription yields (omitting the particle label k)

$$\ddot{\gamma}^\mu = q F^\mu_\nu \dot{\gamma}^\nu. \tag{2}$$

Multiplying both sides by $\dot{\gamma}_\mu$ and using the antisymmetry of F, we get that $\frac{\mathrm{d}}{\mathrm{d}s}\dot{\gamma}^2 = 0$; hence $\dot{\gamma}^2$ is conserved by the s-evolution. This is a direct consequence of the s-independence of the action (1) and can also be expressed as the conservation of a 'mass-squared current':

$${}^k b(x) = \int_{-\infty}^{\infty} \mathrm{d}s \, \delta^{(4)}(x - {}^k\gamma_s) \, {}^k\dot{\gamma}_s^2 \, {}^k\dot{\gamma}_s. \tag{3}$$

Defining $m = \left(\int \mathrm{d}^3\boldsymbol{x}\, b^0\right)^{1/2} \equiv \sqrt{\dot{\gamma}^2} \equiv \frac{\mathrm{d}\tau}{\mathrm{d}s}$ with $\tau = \int^s \sqrt{(\mathrm{d}\gamma)^2}$ the proper time, equation (2) takes the familiar form

$$m\ddot{x}^\mu = q \, F^\mu_\nu \dot{x}^\nu, \tag{4}$$

with $x(\tau) = \gamma(s(\tau))$ above standing for the same world-line parametrized by proper time. We see that the (conserved) effective mass m emerges as a constant of motion associated with a particular solution rather than entering the equations as a fixed parameter. Equation (2), however, is more general than (4) and supports solutions conserving a negative $\dot{\gamma}^2$ (tachyons—whether realistic or not).

Writing the potential entering the square brackets of the action (1) in the form $A({}^k\gamma_s) \equiv \int \mathrm{d}^4 x \delta^{(4)}(x - {}^k\gamma_s) A(x)$, the variation of (1) with respect to A gives Maxwell's inhomogeneous equations

$$\partial_\nu F^{\nu\mu} \equiv \Box A^\mu - \partial^\mu(\partial \cdot A) = \sum_{k=1}^{n} {}^k j^\mu, \tag{5}$$

with

$${}^k j(x) = q \int_{-\infty}^{\infty} \mathrm{d}s \, \delta^{(4)}(x - {}^k\gamma_s) {}^k\dot{\gamma}_s, \tag{6}$$

a conserved current associated with charge k. Equation (5), like any other linear inhomogeneous equation, defines A up to a homogeneous solution of that equation. To uniquely select a solution, the retarded Liénard–Wiechert potential is chosen,

$$A^\mu(x) = q \sum_{k=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}s \, \delta[(x - {}^k\gamma_s)^2] {}^k\dot{\gamma}^\mu \, \theta(x^0 - {}^k\gamma^0), \tag{7}$$

introducing an arrow of time into classical electrodynamics, so manifest in observations.

Translation invariance of the action (1) implies the existence of a conserved energy–momentum tensor which can be chosen to be symmetric:

$$p^{\nu\mu}(x) = \Theta^{\nu\mu}(x) + \sum_{k=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}s\, {}^k\dot{\gamma}^{\nu}\, {}^k\dot{\gamma}^{\mu}\, \delta^{(4)}(x - {}^k\gamma_s), \tag{8}$$

$$\text{with} \qquad \Theta^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F^2 + F^{\nu\rho} F^{\mu}_{\rho}, \tag{9}$$

the (gauge-invariant, symmetric and traceless) canonical radiation energy–momentum tensor, leading to four conserved momenta, $P^{\nu} = \int \mathrm{d}^3\boldsymbol{x}\, p^{\nu 0}$.

Following similar considerations one can get conserved currents associated with Lorentz transformations:

$$J^{\nu\rho,\mu} = p^{\mu\nu} x^{\rho} - p^{\mu\rho} x^{\nu}, \tag{10}$$

with corresponding conserved generalized angular momenta $\int \mathrm{d}^3\boldsymbol{x}\, J^{\nu\rho,0}$ (only six independent ones, due to antisymmetry in ν, ρ). Their conservation need only assume the conservation of $p^{\nu\mu}$ and its symmetry in μ, ν.

The action (1) enjoys the formal scaling symmetry

$$A(x) \mapsto \lambda^{-1} A(\lambda^{-1} x), \qquad \gamma(s) \mapsto \lambda \gamma(\lambda^{-2} s), \tag{11}$$

which is sufficient for a local extremum to be mapped by the scaling transformation to another local extremum (equivalently, the system (2), (5) is invariant under the transformation (11)). Under that transformation, the conserved charge $q = \int \mathrm{d}^3\boldsymbol{x} j^0$ is left unchanged, while the effective mass of a particle scales as $m \mapsto \lambda^{-1} m$. Note that it is the apparently strange scaling power of the parameter s (2 rather than 1 of the proper time) that guarantees this invariance. This auxiliary parameter, having no physical meaning, is eventually eliminated by the integral in (6), leaving an ordinary four-current (or equivalently the world line $\bigcup_s \gamma_s$). Associated with the scaling symmetry (11) is a conserved 'dilatation current'

$$\xi^{\nu} = p^{\nu\mu} x_{\mu} - \sum_{k=1}^{n} \int \mathrm{d}s\, \delta^{(4)}(x - {}^k\gamma_s) s\, {}^k\dot{\gamma}_s^2\, {}^k\dot{\gamma}_s^{\nu}, \tag{12}$$

the proof of whose conservation need only assume the conservation of the energy–momentum tensor (8), the tracelessness of Θ and the conservation of ${}^k\dot{\gamma}_s^2$.

Now, what is the meaning of the conserved dilatation charge

$$D = \int \mathrm{d}^3\boldsymbol{x}\, \xi^0(x^0, \boldsymbol{x})\, ? \tag{13}$$

In analogy to other conserved charges associated with the symmetries of the Poincaré group, D should represent a weighted rate of 'motion in scale' (much like the conserved momenta represent a weighted rate of motion in spacetime, and the conserved angular momentum represents a weighted rate of rotation about a point). It gets a geometric contribution (the $p^{\nu\mu} x_{\mu}$ term benefits from outward motion about the origin, much like the conserved energy benefits from motion in time) and an 'intrinsic' contribution (the counterpart of the binding energy), resulting from matter–radiation interaction.

Unfortunately, there is no simple way to isolate the interesting intrinsic part. The value of D further depends on the choice of origin for both space and the ${}^k s$ parameters of the n trajectories ${}^k\gamma$ (its conservation implies that it is unchanged by a shift in the origin of time). Under a shift of the origin $(\boldsymbol{x}, {}^1 s, \ldots, {}^n s) \mapsto (\boldsymbol{x} + \boldsymbol{a}, {}^1 s + b_1, \ldots, {}^n s + b_n)$, we get

$$D \mapsto D + \boldsymbol{P} \cdot \boldsymbol{a} + \sum_{k=1}^{n} {}^k m^2 b_k. \tag{14}$$

While the dependence of D (as that of the generalized angular momenta) on the choice of origin of space is acceptable (rotation/dilatation around which point?), its dependence on the choice of origin for ${}^k s$ is not. Those n parameters have no physical interpretation whatsoever and are eventually eliminated from any physical prediction through the integral in (6).

Finally, we mention yet another symmetry of the action (1), discrete this time:

$$\gamma(s) \mapsto \gamma(-s), \qquad A(x) \mapsto -A(x). \tag{15}$$

Note that the electric charge (6) changes sign under this 'charge conjugation' symmetry, while the effective mass $\sqrt{\ddot{\gamma}^2}$ is unaltered. In the scale covariant formulation, therefore, the existence of an antiparticle need not be postulated; it is a direct consequence of the symmetry of the action (1).

Despite the matter–radiation coupling term in the action (1), the energy–momentum tensor (8) and the dilatation current (12) (as do the currents associated with Lorentz symmetry) split into a pure EM contribution plus a pure mechanical contribution. This situation does not carry on to ECD. Genuine coupling between 'matter degrees-of-freedom' and those of the EM field appear in the corresponding currents, and even in the counterparts of the electric current (6) and the mass-squared current (3). As the EM field is due to *all* charges, this coupling introduces a novel form of entanglement between charges, the consequences of which we have only begun to explore.

3. Extended charge dynamics (ECD)

3.1. The governing action

In this section, we formulate the ECD equations as the Euler–Lagrange equations of a variational problem. We then give those equations, after dealing with some technical subtleties, the status of a *definition*. Nevertheless, the underlying variational problem, via Noether's theorem, motivates the derivation of many conservation laws associated with symmetries of the variational problem.

The mathematical object characterizing a scalar ECD particle is a pair $\{\phi, \gamma\}$, where $\phi(x, s) : \mathbb{R}^4 \times \mathbb{R} \mapsto \mathbb{C}$ is a complex 'wavefunction' defined over four-dimensional Minkowski space and a scalar s, and $\gamma_s : \mathbb{R} \mapsto \mathbb{R}^4$ is a trajectory in Minkowski space parameterized by s. The dynamics of n ECD particles in interaction with an electromagnetic potential A is governed by the following action. That is, permissible ECD solutions are n pairs $\{\phi, \gamma\}$ and a potential A at which the following real valued functional is stationary:

$$\mathcal{A}[\{{}^1\phi, {}^1\gamma\}, \ldots, \{{}^n\phi, {}^n\gamma\}; A] = \sum_{k=1}^{n} \mathcal{A}_{\mathrm{m}}[\{{}^k\phi, {}^k\gamma\}, A] + \mathcal{A}_{\mathrm{r}}[A], \tag{16}$$

where

$$\mathcal{A}_{\mathrm{r}}[A] = \int_{\mathbb{R}^4} \mathrm{d}^4x \mathcal{L}_{\mathrm{r}}[A] \equiv \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{17}$$

is the classical radiation action with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\begin{aligned}
\mathcal{A}_{\mathrm{m}}[\{\phi, \gamma\}, A] &= \int_{-\infty}^{\infty} \mathrm{d}s \int_{\mathbb{R}^4} \mathrm{d}^4x \mathcal{L}_{\mathrm{m}}[\phi, A] + \frac{2}{\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}s |\phi|^2(\gamma_s, s) \\
&\equiv \int_{-\infty}^{\infty} \mathrm{d}s \int_{\mathbb{R}^4} \mathrm{d}^4x \left[\frac{\mathrm{i}\hbar}{2} (\phi^* \partial_s \phi - \partial_s \phi^* \phi) \right. \\
&\quad \left. - \frac{1}{2} (D^\mu \phi)^* D_\mu \phi + \frac{2}{\mathcal{N}} \delta^{(4)}(x - \gamma_s) \phi^* \phi \right],
\end{aligned} \tag{18}$$

with

$$D_\mu = \bar{h}\partial_\mu - \mathrm{i}qA_\mu, \tag{19}$$

the gauge covariant derivative. Above, q $\bar{h}$ and $\mathcal{N}$ are real constants. A straightforward generalization to higher-spin ϕ's is briefly discussed below. For the sake of simplicity, we shall take $\bar{h} = 1$ unless otherwise indicated.

Ignoring the δ-function term in (18), the governing action (16) is invariant under

$$\phi(x, s) \mapsto \bar{\phi} = \lambda^{-2}\phi(\lambda^{-1}x, \lambda^{-2}s), \qquad A(x) \mapsto \bar{A} = \lambda^{-1}A(\lambda^{-1}x), \tag{20}$$

from which one can read the representation under which each variable (and products thereof) transforms in a scale transformation—its 'scaling dimension' or simply 'dimension': 1 for x; 2 for s; -1 for A; -2 for ϕ; and,by definition, 0 for $q, \bar{h}$ and the omitted speed of light c. The δ-function term, however, spoils that invariance. Under the mandatory scaling $\gamma(s) \mapsto \bar{\gamma} = \lambda\gamma(\lambda^{-2}s)$, we get $\mathcal{A}[\{\phi, \gamma\}, A; \mathcal{N}] = \mathcal{A}[\{\bar{\phi}, \bar{\gamma}\}, \bar{A}; \lambda^{-2}\mathcal{N}]$, sending us to *another* theory, indexed by a transformed parameter $\lambda^{-2}\mathcal{N}$ of dimension -2.

So where hides scale covariance? Obviously, setting $\mathcal{N} = \infty$ in (18) would recover scale covariance, but it would also eliminate γ from the equations. Instead, we define a specific one-parameter family of extrema of (16), indexed by $\mathcal{N}$, and take the (highly nontrivial) limit $\mathcal{N} \to \infty$ involving both ϕ and γ. The ECD formalism therefore has a renormalization-group 'flavor' to it and like the latter, it introduces problematic divergences. However, these divergences are eliminated in a novel way.

3.2. The equations of motion of $\{\phi, \gamma\}$ *('matter')*

To find the necessary conditions for a trial pair $\{\phi, \gamma\}$ to be an extremum of $\mathcal{A}$, we use the standard Euler–Lagrange technique. As A is assumed fixed in this section, we need not consider $\mathcal{A}_\mathrm{r}$, only $\mathcal{A}_\mathrm{m}$. Fixing first the degrees of freedom of ϕ and imposing the vanishing of the first variation of $\mathcal{A}_\mathrm{m}$ with respect to γ^μ, we get at once by the first line of (18),

$$\partial_\mu\, |\phi(\gamma_s, s)|^2 = 0. \tag{21}$$

This equation has, in fact, much in common with the guiding equation of the Bohm–de Broglie pilot-wave theory. Taking the full derivative of (21), we get

$$H_{\nu\mu}(s)\dot{\gamma}^\nu(s) + f_\mu(s) = 0, \tag{22}$$

with $H_{\mu\nu}(s) \equiv \partial_\mu\partial_\nu|\phi(\gamma_s, s)|^2$, the local Hessian matrix, and $f_\mu(s) \equiv \partial_s\partial_\mu|\phi(\gamma_s, s)|^2$. For a given ϕ, therefore, a solution for γ can be obtained by integrating

$$\dot{\gamma} = -H^{-1}f. \tag{23}$$

In that sense, ϕ 'guides' γ and incorporates into its dynamics nonlocal information about distant potentials (such as the status of the 'other slit' in a double slit experiment), much like the Bohm–de Broglie pilot wave does.

Taking the derivative of (23), the right-hand side appears as a generalized Lorentz force derived from the local morphology of $|\phi(x, s)|$ at $x = \gamma_s$ (indeed, we shall prove that it is orthogonal to $\dot{\gamma}_s$). In fact, in appendix B, we show that for an external EM field, F, smoothly varying over the charge's Compton length, it is just the Lorentz force (2). However, when the scale of variation of F competes with the Compton length, the dynamics may become highly non-classical and possibly violent when H becomes singular. That singularity generically has nothing to do with the strength of F, and it may lead to violent behavior even in a weak field, which, we speculate, is manifested in Compton scattering and the photoelectric effect.

Next, fixing γ and imposing the vanishing of the first variation with respect to ϕ, we get

$$\left[\mathrm{i}\partial_s - \mathcal{H} + \frac{2}{\mathcal{N}}\delta^{(4)}\left(x - \gamma_s\right)\right]\phi(x, s) = 0, \tag{24}$$

with the Hamiltonian given by

$$\mathcal{H} = -\tfrac{1}{2}D^\mu D_\mu. \tag{25}$$

This equation, involving a distribution, needs to be interpreted and handled very carefully. Symbolically, we see that γ enters a proper time Schrödinger equation not as a source, but rather as an s-dependent 'delta-function potential' supported on the instantaneous position of the particle in Minkowski space at proper time s. Combined with (21), it turns the master–slave relationship between the wave and the point, that is present in the Bohm-de Broglie theory, into a symbiotic one.

Obviously, interpreting this equation (and, even more so, solving it,) is very difficult, so we recast it in an integral form specifying the role of the delta-function potential. Introducing a positive relaxation parameter, ϵ, which will later be taken to zero, yields

$$\begin{aligned}\phi(x, s) = {}& \frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{s-\epsilon} \mathrm{d}s'\, G(x, \gamma_{s'}; s - s')\phi(\gamma_{s'}, s') \\ & - \frac{\mathrm{i}}{\mathcal{N}}\int_{s+\epsilon}^{\infty} \mathrm{d}s'\, G(x, \gamma_{s'}; s - s')\phi(\gamma_{s'}, s') \\ \equiv {}& \frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty} \mathrm{d}s'\, G(x, \gamma_{s'}; s - s')\phi(\gamma_{s'}, s')\mathcal{U}(\epsilon; s - s'),\end{aligned} \tag{26}$$

$$\text{with} \qquad \mathcal{U}(\epsilon; \sigma) = \theta(\sigma - \epsilon) - \theta(-\sigma - \epsilon). \tag{27}$$

Here, G is the *propagator*, a solution to the homogeneous (proper time) Schrödinger equation[1] (also known as a five-dimensional Schrödinger equation, discussed in appendix B),

$$[\mathrm{i}\partial_s - \mathcal{H}]\, G(x, x'; s) = 0, \tag{28}$$

satisfying the initial condition (in the distributional sense)

$$G(x, x'; s) \underset{s\to 0}{\longrightarrow} \delta^{(4)}(x - x'). \tag{29}$$

The formal equivalence of (24) and (26) is established by operating with $\mathrm{i}\partial_s - \mathcal{H}$ on both sides of (26), taking the limit $\epsilon \to +0$, and using

$$\delta^{(4)}(x - x')f(x') = \delta^{(4)}(x - x')f(x) \quad \forall f.$$

Note that the set of solutions of (26) forms a vector space, as expected of the original linear differential equation (24).

To avoid possible pitfalls often accompanying formal manipulations such as the above, we shall regard the system (21) and (26) as a *definition*, dubbed the 'central ECD system'.

[1] This equation, a variant of which was first discovered by Fock, shares with the non-relativistic Schrödinger equation all the good properties to which the latter owes its fame. In particular, the modulus-squared of its solutions is a positive quantity conserved by the s-evolution and hence can be regarded as a spacetime density. Furthermore, by Ehrenfest's theorem, localized wave-packets satisfy the relativistic classical EOM. The reason that it is not used directly in relativistic QM has to do with the interpretation of the parameter s. In the classical theory, s only serves to parameterize the world line (and can therefore be eliminated from any physical prediction). Such direct elimination obviously cannot be extended to the quantum case. This is why variants of this equation make their appearance in the literature only as an intermediate step in the calculations of s-independent quantities, either by taking its steady state solutions or by integrating its solutions over s. An exception to this statement is the interpretation of Stückelberg and later Horwitz (see [1] and references therein) postulating the existence of a 'universal' (frame independent) time played by s.

Gauge invariance of the central ECD system is established by the transformation law for the propagator

$$A \mapsto A' = A + \partial\alpha \quad \Rightarrow \quad G \mapsto G' = G \exp \mathrm{i}q[\alpha(x) - \alpha(x')]. \tag{30}$$

It then follows that if $\{\phi, \gamma\}$ is a solution of the system (21) and (26), with potential A, then $\{\phi \exp \mathrm{i}q\alpha, \gamma\}$ is a solution with the gauge transformed potential A'.

Finally, we mention a simple generalization of the central ECD system for particles whose ϕ part transforms under a nontrivial representation of the Lorentz group. For example, when ϕ is a four-component bispinor, the propagator G propagates a bispinor wavefunction satisfying

$$\mathrm{i}\partial_s\phi = \left(\mathcal{H} + \frac{q}{2}\sigma_{\mu\nu}F^{\mu\nu}(x)\right)\phi, \tag{31}$$

where $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$, with γ_μ a Dirac matrix (not to be confused with γ the trajectory). The initial condition (29) at $s = 0$ is now the unit operator in spinor-space. Now, $\phi^*\phi$ in (21) should read $\overline{\phi}\phi$, with $\overline{(\cdot)} = (\cdot)^\dagger\gamma^0$ (γ^0 being the Dirac matrix diag$(1, 1, -1, -1)$), which is a real quantity but is not positive definite. This poses no problem, as no probabilistic interpretation is attributed to it (nor to $\phi^*\phi$ for that matter.) Similarly, in the Lagrangian (18) (and in the rest of the paper): $(\cdot)^* \mapsto \overline{(\cdot)}$ and $(D\phi)^* \cdot D\phi \mapsto \overline{\not{D}\phi}\not{D}\phi$ with $\not{D} \equiv D^\mu\gamma_\mu$.

3.2.1. Example: a free particle. There are obviously no standard mathematical tools for finding a general solution to the central ECD system (21) and (26). Nevertheless, verification of a guess may be straightforward. Consider then a freely moving particle in a field-free spacetime. Gauge invariance implies that we can assume $A = 0$ throughout the spacetime. The Hamiltonian (25) is then just the free Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\,\Box, \tag{32}$$

and the corresponding propagator can be computed using a variety of methods[2] yielding

$$G_{\mathrm{f}}(x, x'; s) = \frac{\mathrm{i}}{4\pi^2}\frac{\mathrm{e}^{\frac{\mathrm{i}(x-x')^2}{2s}}}{s^2}\,\mathrm{sign}(s). \tag{33}$$

Next, we show that a freely moving particle constitutes a solution to the system (21) and (26), that is, the ECD pair has $\gamma_s = us$ for its trajectory part. The constant velocity u is assumed time-like, i.e. $u^2 > 0$, although tachyons—with space-like u's—can equally be treated.

We can now attempt to compute the integral in (26). Using the ansatz

$$\phi\,(\gamma_s, s) = C \exp \mathrm{i}\frac{u^2}{2}s \tag{34}$$

in (26) with C some complex constant of dimension -2, we substitute $x' \mapsto us'$, shift the integration variable $s - s' \mapsto -s'$ and define $\xi := x - us$, obtaining

$$\phi(x, s) = \frac{-C}{4\pi^2\mathcal{N}}\,\mathrm{e}^{\mathrm{i}(u\cdot\xi + \frac{u^2}{2}s)}\left(\int_{-\infty}^{-\epsilon}\mathrm{d}s'\,\frac{\exp\frac{-\mathrm{i}\xi^2}{2s'}}{s'^2} + \int_{\epsilon}^{\infty}\mathrm{d}s'\,\frac{\exp\frac{-\mathrm{i}\xi^2}{2s'}}{s'^2}\right). \tag{35}$$

To see that (26) is satisfied, we need only check the consistency condition $\phi(us, s) = C \exp \mathrm{i}\frac{u^2}{2}s$. Plugging $x \mapsto us$ above ($\xi = 0$), we get

$$C \exp \mathrm{i}\frac{u^2}{2}s = \frac{-C \exp \mathrm{i}\frac{u^2}{2}s}{4\pi^2\mathcal{N}}\left(\int_{-\infty}^{-\epsilon}\mathrm{d}s'\,\frac{1}{s'^2} + \int_{\epsilon}^{\infty}\mathrm{d}s'\,\frac{1}{s'^2}\right), \tag{36}$$

[2] One method is by the Laplace transform with respect to s; another is to regard $\mathcal{H}$ as a sum of four commuting operators, so that the propagator factorizes into a product of four one-dimensional propagators, which in turn reduces to a Gaussian integral; Schwinger computes it using Heisenberg's EOM [4].

implying that (26) is satisfied provided that

$$\mathcal{N} = \frac{-1}{4\pi^2}\left(\int_{-\infty}^{-\epsilon} \mathrm{d}s' \, \frac{1}{s'^2} + \int_{\epsilon}^{\infty} \mathrm{d}s' \, \frac{1}{s'^2}\right) = -\frac{1}{2\pi^2\epsilon}, \tag{37}$$

in which case any $C \in \mathbb{C}$ is a solution. This freedom in the scale of ϕ, i.e.

$$\phi \mapsto C\phi, \tag{38}$$

is an exact symmetry of the central ECD system, as is straightforwardly verified. To further give ϕ the right dimension and prevent it from vanishing in the $\epsilon \to 0$ limit, we must choose $C \propto \epsilon^{-1}$.

With the above relation between $\mathcal{N}$ and ϵ, (35) now holds for any x, and we get

$$\phi(x,s) = \frac{-C}{2\pi^2\mathcal{N}} \, \mathrm{e}^{\mathrm{i}(u\cdot\xi + \frac{u^2}{2}s)} \frac{1}{\epsilon} \mathrm{sinc}\left(\frac{\xi^2}{2\epsilon}\right), \qquad \text{with} \quad \mathrm{sinc}(y) \equiv \frac{\sin y}{y}. \tag{39}$$

Using $\lim_{\epsilon\to 0} \epsilon^{-1}\mathrm{sinc}(\epsilon^{-1}x) \propto \delta(x)$ (in fact, any other integrable function other than sinc would also have this property) and $\Box\,\delta(\xi^2) = \delta(\xi)$, the above form of ϕ can explicitly be shown to satisfy Schrödinger's equation (28) for $\xi \neq 0$. However, the form (39) satisfies (26) *everywhere* also for a finite ϵ, which is a crucial part of the formalism.

To check that the other ECD equation—equation (21)—is satisfied by the free ϕ, we first write it in an alternative form:

$$\mathrm{Re}\ \partial_\mu \phi(\gamma_s, s)\phi^*(\gamma_s, s) = 0. \tag{40}$$

This form has an illuminating interpretation: identifying a complex number $z \equiv a + b\,\mathrm{i} \in \mathbb{C}$ with the vector $(a, b) \in \mathbb{R}^2$, equation (40) states that the vectors in $\mathbb{R}^2$ associated with $\partial_\mu\phi(\gamma_s, s)$ and $\phi(\gamma_s, s)$ are orthogonal with respect to the standard inner product in $\mathbb{R}^2$. Introducing the explicit form (33) of the free propagator into (26), and exchanging the order of differentiation and integration, we get

$$\begin{aligned}
\partial_\mu \phi(us, s) = \frac{\mathrm{i}}{\mathcal{N}}\Bigg(&\int_{-\infty}^{s-\epsilon} \mathrm{d}s' \, \partial_\mu \frac{\mathrm{i}(x - us')^2}{2(s-s')}\bigg|_{x=us} G_{\mathrm{f}}(us, us'; s-s')\phi(us', s') \\
&- \int_{s+\epsilon}^{\infty} \mathrm{d}s' \, \partial_\mu \frac{\mathrm{i}(x-us')^2}{2(s-s')}\bigg|_{x=us} G_{\mathrm{f}}(us, us'; s-s')\phi(us', s')\Bigg) \\
= \mathrm{i}u_\mu \phi(\gamma_s, s),&
\end{aligned} \tag{41}$$

establishing the orthogonality condition (40).

3.3. *The equations of motion of A ('radiation')*

Advancing now to the variational problem associated with the potential field A in (16), we find by the Euler–Lagrange equations that

$$\partial_\nu F^{\nu\mu} \equiv \Box A^\mu - \partial^\mu(\partial\cdot A) = \sum_{k=1}^{n} {}^k j^\mu, \tag{42}$$

with ${}^k j$ a gauge invariant current of dimension -3 associated with the kth particle

$${}^k j^\mu = \int_{-\infty}^{\infty} \mathrm{d}s \, \frac{\mathrm{i}q}{2}[{}^k\phi(D^{\mu\,k}\phi)^* - {}^k\phi^* D^{\mu\,k}\phi] \equiv \int_{-\infty}^{\infty} \mathrm{d}s \, q \, \mathrm{Im}{}^k\phi^* D^{\mu\,k}\phi. \tag{43}$$

Equation (42) is just the inhomogeneous Maxwell's equation, with $j \equiv \sum_{k=1}^n {}^k j^\mu$ a source. However, as A appears also on the right-hand side of (43) (through D's dependence on it), equations (42) and (43), unlike their classical counterparts, are not a *prescription* for generating a potential given the DOF of all matter (${}^k\gamma$ in the classical case; $\{{}^k\phi, {}^k\gamma\}$ in our case), but rather an equation coupling the two. This unusual coupling removes the arbitrariness in the choice of a homogeneous solution to Maxwell's equations which can be added to A, as in the classical case, and, as a result, excludes a simple introduction of an arrow of time such as the one resulting from the choice of the retarded Liénard–Wiechart potential.

It turns out that expression (43) diverges everywhere in the limit $\epsilon \to 0$, and it is hence useless without further calibrations. The origin of the divergence is the modulus squared of the light-cone singularity of $\phi(x, s)$ on $(x - \gamma_s)^2 = 0$, already encountered in section 3.2.1. Formally, expression (43) for the current, being bilinear in ϕ and ϕ^*, contains a meaningless product of two $\delta(\xi^2)$ distributions. (This pathology is also encountered in QFT, necessitating the renormalization procedure.) Nevertheless, the divergences in ECD can be shown to be entirely associated with γ and are unambiguously removed, as demonstrated by the next example.

3.3.1. The current of a free charge to first order in q. An exact steady-state solution of the ECD equations is beyond the scope of this paper, as $\{\phi, \gamma\}$ and A are genuinely coupled by the central ECD system (21) and (26) in one direction and by (42) in the reverse direction. This coupling severely restricts the set of solutions, possibly leading to quantization of charge which is scale invariant; see section 3.4.

A perturbative expansion of the current in q (assumed small) is one way of coping with the problem. As the divergences already appear at first order, and are entirely due to the s^{-2} divergence of the propagator—a universal, potential-independent divergence—the divergence issue is fully covered by analysis of the current associated with ϕ in (39).

Plugging (39) into expression (43) for the current (with $A = 0$), we get the spherically symmetric current

$$j^0(r) = \frac{q|C|^2}{4\pi^4|\mathcal{N}|^2} u^0 \int_{-\infty}^{\infty} \mathrm{d}s \, \frac{1}{\epsilon^2} \mathrm{sinc}^2\left(\frac{\xi^2}{2\epsilon}\right) \underset{\epsilon\to 0}{\longrightarrow} \frac{q|C|^2 \,\mathrm{sign}\, u^0}{4\pi^4|\mathcal{N}|^2} \frac{1}{\epsilon r}. \tag{44}$$

To cope with the ϵ^{-1} divergence of j^0 we note that in the limit $\epsilon \to 0$, the current (44) is due entirely to the singularity of $\phi(x, s)$ on the light-cone of γ_s and generalizes to the conserved current

$$j^\mu_{\mathrm{div}}(x) = \frac{q|C|^2}{4\pi^4|\mathcal{N}|^2\epsilon} \int_{-\infty}^{\infty} \mathrm{d}s \, \delta[(x - \gamma_s)^2]\dot{\gamma}^\mu_s. \tag{45}$$

This divergent term can therefore be subtracted from (43), leaving a well behaved conserved current.

Returning to the analysis of the integral (44), we note that on dimensional grounds it must have the form $\epsilon^{-3/2} f(\epsilon^{-1/2} r)$. As noted, the large x behavior of $f(x)$ is given by x^{-1}. By systematically accounting for the the non-vanishing width of $\mathrm{sinc}(\xi^2/\epsilon)$ and its symmetry, higher power corrections can be computed, with x^{-5} turning out to be the leading order. We see, then, that (a) f is everywhere finite for any $\epsilon > 0$ and (b) that after removal of the x^{-1} term, it is integrable in three dimensions. Combined with its scaling law, we can therefore deduce that $\lim_{\epsilon\to 0} j^0(\boldsymbol{x}) \propto \delta^{(3)}(\boldsymbol{x})$. Note that as this δ-distribution is a result of a definite limit procedure, all infinities associated with it (such as the divergent self energy; see section 3.4 below) can, in principle, be separated from the interesting finite part[3].

[3] A δ-function distribution is, in fact, the only integrable scale invariant distribution, ϵ and u—the only dimensional parameters in the problem—having been eliminated. As an external EM field introduces another length scale, this singular nature of the finite part of the current is a peculiarity of the free solution.

3.4. Symmetries and conservation laws

The existence of an action principle for ECD suggests that there are conservations laws associated with the symmetries of the action (16). The connection between the two is established via Noether's theorem and is the content of this section. As our action principle was only meant to motivate the equations of motion, in appendix A we give an independent proof of the conservation of the electric current, based directly on the central ECD system (21) and (26), which easily generalizes to the conservation of all other currents.

It is instructive to compare the ECD currents with their classical counterparts derived in section 2, the most significant differences being the nonsingular nature of that part of the current associated with the matter DOF, and the highly entangled form of radiation and matter, not present in the classical currents.

There are two kinds of conservation laws in ECD. One concerns quantities that are conserved separately for each particle, such as charge and mass, and the other involves all the DOF of the combined system of particles plus EM potential, e.g. energy, momentum and angular momentum. We begin with charge conservation. The $U(1)$ symmetry of each of the n ECD actions (18) ($\phi \mapsto e^{i\alpha}\phi$, with a constant real α) implies the existence of n conserved currents via Noether's theorem. We use the standard trick in deriving that conserved current, that is, we shift an extremal pair $\{\phi, \gamma\} \mapsto \{e^{iq\Lambda}\phi, \gamma\}$ with an infinitesimal $\Lambda(x, s)$ vanishing for $|x_\mu|, |s| \to \infty$ and impose the vanishing of the variation to first order in Λ. Using the global $U(1)$ symmetry, and the fact that the $\frac{2}{\mathcal{N}}\delta^{(4)}(x - \gamma_s)\,\phi\phi^*$ term is unaffected by gauge transformations, we arrive at the usual continuity equation associated with the homogeneous Schrödinger equation (28):

$$\partial_s \rho + \partial_\mu J^\mu = 0,$$
$$\text{with} \qquad \rho = q\phi\phi^* \qquad \text{and} \qquad J^\mu = q\,\mathrm{Im}\,\phi^* D^\mu \phi. \tag{46}$$

Integrating (46) with respect to s from $-\infty$ to ∞ (we can safely assume $\phi(x, s) \to 0$ as $|s| \to \infty, \forall x$), we get

$$\partial_\mu j^\mu = 0, \tag{47}$$

with j the current (43). (Had our Λ depended on x alone, we could have obtained (47) directly from the requirement of stationary action. Further note our convention to lower the case of a letter after integration with respect to s.) Further, (47) implies that the monopole field of an ECD particle is one and the same even when the particle's motion is highly non-classical, such as that of a bound electron.

The next symmetry applying to the individual particles is s-translation. This time we infinitesimally shift the s coordinate of an extremal $\phi(x, s)$, $s \mapsto s + \Lambda(x)$. The vanishing of the first variation entails

$$\partial_\mu \int_{-\infty}^{\infty} \mathrm{d}s\, \bar{B}^\mu + \frac{2}{\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}s\, \partial_s |\phi|^2(x, s)\, \delta^{(4)}(x - \gamma_s) = 0, \tag{48}$$

$$\text{with} \qquad \bar{B}^\mu = \mathrm{Re}\,\partial_s\phi^* D^\mu \phi \quad \text{or} \quad -\mathrm{Re}\,\phi^* \partial_s D^\mu \phi, \tag{49}$$

after integration by parts. The second term in (48) is also a divergence

$$\begin{aligned}
\partial_\mu \breve{b}^\mu &\equiv \partial_\mu \int_{-\infty}^{\infty} \mathrm{d}s\, \breve{B}^\mu = \partial_\mu \int_{-\infty}^{\infty} \mathrm{d}s \left[\frac{2}{\mathcal{N}} |\phi|^2(x, s) \delta^{(4)}(x - \gamma_s) \dot{\gamma}_s^\mu \right] \\
&= \frac{2}{\mathcal{N}} \left[\int_{-\infty}^{\infty} \mathrm{d}s\, \partial_\mu |\phi|^2(x, s) \delta^{(4)}(x - \gamma_s) \dot{\gamma}_s^\mu + \int_{-\infty}^{\infty} \mathrm{d}s |\phi|^2(x, s) \partial_\mu \delta^{(4)}(x - \gamma_s) \dot{\gamma}_s^\mu \right] \\
&= -\frac{2}{\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}s |\phi|^2(x, s) \frac{\mathrm{d}}{\mathrm{d}s} \delta^{(4)}(x - \gamma_s) = \frac{2}{\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}s\, \partial_s |\phi|^2(x, s) \delta^{(4)}(x - \gamma_s).
\end{aligned} \tag{50}$$

The reason we can ignore the first term in the second line, as far as conservation laws are concerned, is the following. To get a conserved quantity we integrate (48) over a volume in spacetime and apply the Stokes theorem to get a conserved flux. Now, the integral of that first term over any volume vanishes by (21). We see that

$$b \equiv \bar{b} + \breve{b} = \int_{-\infty}^{\infty} \bar{B} + \breve{B}\, \mathrm{d}s \tag{51}$$

is therefore conserved. It has dimension −5 and can be computed for the free case to zeroth order in q—the counterpart of the electric current computed in section 3.3.1. Its divergent piece (the counterpart of j_{div}) has a more complex form which, like j_{div}, is uniquely determined by γ and can be shown to be conserved independently of the form of γ, with the exception of a part which reads

$$b^{\mu}_{\mathrm{div}}(x) = \frac{|C|^2}{4\pi^4|\mathcal{N}|^2\epsilon} \int_{-\infty}^{\infty} \mathrm{d}s\, \delta[(x-\gamma_s)^2]\dot{\gamma}_s^2\dot{\gamma}_s^{\mu}, \tag{52}$$

which is conserved only if $\dot{\gamma}_s^2$ is s-independent, proving that, as in classical electrodynamics, γ remains on a fixed mass shell.

The finite b surviving the removal of the trivial divergent piece is proportional to r^{-5}—the only scale invariant current of dimension −5. It demonstrates that even at zeroth order, the ‘extent of an ECD charge’ goes far beyond its center where the charge is concentrated. We need not be alarmed by the nonintegrable singularity of that current. For any finite ϵ the regular current is everywhere finite and integrable, facilitating the isolation of the divergent part of the charge from its interesting finite part.

We move now to the second class of conservation laws, involving all the DOF of a system comprising n ECD pairs $\{\phi, \gamma\}$ and an EM potential A. Let us begin with translational invariance, manifest in the action (16). This is just an Abelian subgroup of symmetries of the full Poincaré group, the derivation of associated conserved currents of which follows similar lines. We shall employ a standard trick (see for example [2, p 22]) in deriving the conserved currents associated with translation invariance. Infinitesimally shifting the argument of $\phi(x,s)$, $x^{\mu} \mapsto x^{\mu} + a^{\mu}(x)$, in an extremal ECD pair $\{{}^k\phi, {}^k\gamma\}$, leaving all other DOF unchanged and imposing that the variation of the action (16) vanishes to first order in a^{μ}, results (after some integrations by part) in

$$\delta\mathcal{A} = \int_{\mathbb{R}^4} \mathrm{d}^4x \int_{-\infty}^{\infty} \mathrm{d}s \left[\partial_{\nu}{}^kT^{\nu\mu} - {}^kV^{\mu} + \frac{2}{\mathcal{N}}\delta^{(4)}(x - {}^k\gamma_s)\partial^{\mu}({}^k\phi\, {}^k\phi^*)\right] a_{\mu}(x) = 0, \tag{53}$$

$$\text{with} \qquad T^{\nu\mu} = g^{\nu\mu}\mathcal{L}_{\mathrm{m}} - \left(\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial(\partial_{\nu}\phi)}\partial^{\mu}\phi + \text{c.c.}\right) \tag{54}$$

$$\text{and} \qquad V^{\mu} = \frac{\partial\mathcal{L}_{\mathrm{m}}}{\partial A_{\lambda}}\partial^{\mu}A_{\lambda}, \tag{55}$$

where $\mathcal{L}_{\mathrm{m}}$ is defined in (18). Preforming the x integrals in (53), the contribution of the last term in the square brackets vanishes by (21). As a is arbitrary, this implies that the rest of the integrand must vanish identically in x. Integrating the remaining two terms over s, we therefore get

$$\partial_{\nu}{}^kt^{\nu\mu} - {}^kv^{\mu} = 0, \tag{56}$$

$$\text{where} \qquad t^{\nu\mu}(x) = \int_{-\infty}^{\infty} \mathrm{d}s\, T^{\nu\mu}(x,s) \tag{57}$$

$$\text{and} \qquad v^\mu = \int_{-\infty}^{\infty} \mathrm{d}s\, V^\mu. \tag{58}$$

As could have been anticipated, the energy–momentum tensor ${}^k t$ is not conserved, as apparent from (56), due to broken translations invariance induced by the potential A, nor is it gauge invariant. To get a conserved quantity we repeat the preceding procedure with A entering (16) resulting in

$$\partial^\nu \bar{\Theta}^{\nu\mu} + \sum_{k=1}^{n} {}^k v = 0, \tag{59}$$

$$\text{with} \qquad \bar{\Theta}^{\nu\mu} = g^{\mu\nu}\mathcal{L}_{\mathrm{r}} - F^\nu_\lambda \partial^\mu A^\lambda. \tag{60}$$

The combination $\bar{\Theta}^{\nu\mu} + \sum_{k=1}^{n} {}^k t$ is therefore conserved. As is customary in classical electrodynamics (see e.g. [2, p 25]) we add to each of the above four conserved currents ($\mu = 0 \ldots 3$) a conserved current not contributing to the conserved charge which renders the above sum symmetric and gauge invariant. In our case this term reads $\partial_\lambda(F^{\nu\lambda}A_\mu)$ which, together with (57), (58) and (42), finally gives a conserved gauge invariant and symmetric energy–momentum tensor

$$p^{\nu\mu} = \Theta^{\nu\mu} + \sum_{k=1}^{n} {}^k m^{\nu\mu}, \tag{61}$$

$$\text{with} \qquad m^{\nu\mu} = \int_{-\infty}^{\infty} g^{\nu\mu}\mathcal{L}_{\mathrm{m}} - \frac{1}{2}(D^\nu\phi(D^\mu\phi)^* + \text{c.c.})\, \mathrm{d}s, \tag{62}$$

$$\text{and} \qquad \Theta^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F^2 + F^{\nu\rho}F^\mu_\rho \tag{63}$$

the canonical EM energy–momentum tensor.

Yet another symmetry involving all the DOF is scale invariance

$$\phi(x,s) \mapsto \lambda^{-2}\phi(\lambda^{-1}x, \lambda^{-2}s), \quad \gamma(s) \mapsto \lambda\gamma(\lambda^{-2}s), \quad A(x) \mapsto \lambda^{-1}A(\lambda^{-1}x), \tag{64}$$

with $\lambda \in \mathbb{R}^+$. The covariance of the central ECD system (21) and (26) under (64) follows from the form of the propagator in the transformed potential ${}^\lambda G(x, x'; s) = \lambda^{-4} G(\lambda^{-1}x, \lambda^{-1}x'; \lambda^{-2}s)$, plus an additional scaling of the parameter $\epsilon \mapsto \lambda^2\epsilon$ (or equivalently $\mathcal{N} \mapsto \lambda^{-2}\mathcal{N}$). The transformed parameter apparently shifts our solution to another theory, indexed by a different parameter. Yet it is only the limit $\epsilon \to +0$ we are interested in, which is equivalent to the $(\lambda^2\epsilon) \to +0$ limit $\forall\lambda > 0$.

Caution must be practiced when trying to derive the associated conserved current via Noether's theorem, as the action (16) is not invariant under (64) without further scaling of $\mathcal{N}$. However, the common procedure of infinitesimally scaling the arguments $x \mapsto \lambda(x)x$, $s \mapsto \lambda^2(x)s$, and imposing the variation of the action to first order in $\ln\lambda$ does lead to a conserved current. Following a symmetrization procedure similar to the one used in the case of the energy–momentum tensor (61), we arrive at a conserved current

$$\xi^\mu = p^{\mu\nu}x_\nu - \sum_k 2\int_{-\infty}^{\infty} \mathrm{d}s\, s({}^k\bar{B}^\mu + {}^k\breve{B}^\mu), \tag{65}$$

where $\bar{B}$ and $\breve{B}$ are defined in (49) and (50), respectively. (Apart from the factor 2—a matter of convention—note the similarity to its classical counterpart (12).)

To conclude this section we mention another symmetry, discrete this time, which therefore has no conserved current associated with it. This symmetry, dubbed 'charge conjugation' or 's-reversal', reads

$$\phi(x,s) \mapsto \phi^*(x,-s), \qquad \gamma(s) \mapsto \gamma(-s), \qquad A(x) \mapsto -A(x). \tag{66}$$

Note that the electric current (43) reverses its sign under this transformation; yet the world line traced by $\gamma(-s)$ is the same as that traced by $\gamma(s)$. This is exactly what is expected of a classical 'anti-particle': the trajectory of a particle in an electromagnetic field A is the same as that of its antiparticle's in $-A$, and the radiations they produce are of opposite sign. This behavior obviously persists in any complex particle, i.e. an aggregate of elementary charges and 'anti-charges' (as the charge of a complex particle is just the sum of charges of its constituents) and also in dynamics outside the classical regime.

4. Discussion

Being scale free, ECD may possibly find applications in a wide variety fields. We next briefly list a few of them.

As a hidden variables model for quantum mechanics, ECD is not expected to yield new predictions—at least not when QM is used as a statistical tool. Even if ECD is indeed the underlying ontology behind QM measurements, QM has an autonomous status of a fundamental law of nature. Trying to derive the statistical results of QM from ECD, in a manner similar to the way thermodynamics is derived from classical mechanics, would necessitate an enormous set of postulates, the counterparts of the ergodicity postulate, and would logically be equivalent to directly accepting QM as a postulate. (For example, what would constitute a natural distribution of the DOF associated with ϕ, having their value in some abstract, non-physical space?) However, the symmetries of ECD, along with their associated conservation laws and its classical limit, severely constrain *any* statistical description of ensembles of it, and it is not difficult to show that relativistic QM satisfies these constraints. Proving the uniqueness of QM (or a class of QM like theories) in this context would therefore be a first step in deriving QM from ECD, the other necessary component being that of compatibility of ECD with QM, namely whether *any* quantum mechanical experiment can be seen as a realization of some reasonable ensemble of ECD solutions. As a counterexample, classical mechanics is manifestly incompatible with QM. On that issue we have made some progress to be reported in a future publication. In this respect, the three relevant ingredients of ECD demonstrated in the present paper are:

(a) the highly non-local nature of the central ECD system, as opposed to classical dynamics;
(b) the possible violent behavior of a charge even in a weak EM field—provided that the latter is rapidly varying—as explained following (23) and
(c) the existence of conservation laws constraining all charges plus the EM field.

In support of the conjectured relevance of ECD, let us briefly show how the illusion of a photon emerges in this framework classically treating the EM field. Consider, then, a small wave packet of total (classical) energy E, centered at a frequency ω, evenly splitting[4] in a beam-splitter and arriving at two photodetectors. A photodetector is a device utilizing the photoelectric effect. Its fine structure invalidates the classical approximation of ECD and by (b) above allows for an electron to be violently ejected as a result of even the smallest perturbation, even if the latter is smoothly varying over the electron's Compton length. By (a),

[4] The evenness assumption of the splitting is for simplicity. In fact, all one can tell is that *on average* the beam is evenly split.

two electrons, one in each detector, interacting with two identical halves of the wave-packet need not respond in the same way, as they have different histories. Likewise, consecutive responses of one electron exhibit a certain distribution—in their final energies, for example—which, as mentioned, is an independent law of nature that is not necessarily derived from ECD. (However, it might be, as indicated by the discussion on plane waves below.) We can consistently assume that the distribution of the final energy of an ejected electron is sharply centered at $g\omega$ for some g (neglecting the binding energy for simplicity). Now, the crucial point is this: although those marginal distributions in energy for each of the two electrons sitting in the two detectors are identical and narrow, their joint distribution does not factorize—the two ejection processes are not statistically independent, as they are constrained by energy conservation (c). It follows that if $g\omega < E < 2g\omega$, one and only one detector can fire, as if the 'photon' has chosen one path only. (Recombining the two halves in a second beam-splitter, as in a Mach–Zehnder interferometer, gives rather prosaic results in light of this interpretation). Taking also linear and angular momentum conservation into account, and using properties (a)–(c), the above arguments can be extended to explain the full range of experiments involving photons within a purely classical treatment of the EM field.

Historically, generating statistical predictions was the sole scope of QM, but gradually, additional more 'deterministic' uses of QM and related concepts emerged. These new applications generally deal with the question of 'what stuff is made of' and what its properties are. For example, the strength of the chemical bonds holding a crystal, or its shape, is not a statistical issue, yet quantum mechanical concepts are at the heart of the chemistry which addresses these questions. High-energy physics is yet another example. The masses of various subatomic particles is, again, not a probabilistic question. Nevertheless, QM, along with its built-in unitarity and the theory of measurement, is fundamentally a statistical tool, whose central role in the realm of certainty might be due to a lack of an alternative: there are no bound states in classical electrodynamics from which matter can be composed.

We speculate that ECD, with its highly non-classical short-distance behavior, may eventually be found to be relevant in such territories, possibly eliminating the need for additional forces and a plethora of subatomic fundamental particles and constants, building only on bound aggregates of elementary ECD charges. That, of course, would require advancing our calculational capabilities (either analytic or numerical) far beyond their status in this paper. As a first concrete exercise we intend to pursue the solution of a free isolated charge from section 3.3.1 to higher orders in q, or even nonperturbatively, in an attempt to discover if some quantization emerges. Note that as the electric charge of a solution is a scale-scalar, any such quantization would imply a universal quantization of the electric charge.

A somewhat simpler place to begin would be to remain at first order in q, and to solve the central ECD system for physically interesting external potentials. Toward this goal we are able to show that the propagator for a scalar ECD charge in a plane wave (of any spectral content) is just the free propagator (33) with the phase replaced by the classical action in that wave. By the results of appendix B, this implies that the γ part of an ECD solution is *exactly* a solution of the corresponding classical EOM. Remarkably, the exactness of the classical solution holds also for arbitrary spin. The calculation of the associated currents, such as the electric current, reduces then to a (cumbersome) integral from which, for example, the cross section for scattering of a plane wave can be derived.

We next consider another possible application of ECD, pertaining rather to large scales. As we have seen, an ECD particle is naturally equipped with a 'remote-sensing' mechanism. This mechanism, however, causes only marginal corrections to classical paths and is only detectable in a scattering experiment, for example, via the large amplification brought about by the huge (relative to the size of the scatterer) distance to the detection screen. But note that there exists another way to amplify those marginal deviations from classical paths, namely the huge numbers involved in gravitational effects. Gravitational and quantum mechanical corrections to classical electrodynamics, therefore, possibly refer to the same ECD 'remote sensing' mechanism, differentiated by two vastly different types of experiments.

Finally, if a conservation law associated with scale covariance truly exists, it is most likely to have an observational signature. Is it possible that the (statistical) Hubble law is just a manifestation of a uniform drift in the scale of the universe as a whole, light arriving from a distant galaxy being emitted at an earlier epoch of larger scale, hence red shifted? (Note that a constant rate in the scale-drift, combined with average homogeneity, implies a linear Hubble relation. Further note the distinction between expansion/contraction and drift in scale, having an intrinsic component.) Such intriguing speculations obviously require a great deal of investigation.

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Appendix A. Current conservation

To prove that the current (43) is conserved in the limit $\epsilon \to 0$, we first need the following lemma, whose proof is obtained by direct computation.

Lemma. *Let $f(x,s)$ and $g(x,s)$ be any (not necessarily square integrable) two solutions of the homogeneous Schrödinger equation (28); then*

$$\frac{\partial}{\partial s}(fg^*) = \partial_\mu\left[\frac{\mathrm{i}}{2}\big(D^\mu fg^* - (D^\mu g)^* f\big)\right]. \tag{A.1}$$

This lemma is just a differential manifestation of unitarity of the Schrödinger evolution—hence the divergence.

Turning now to equation (26),

$$\phi(x,s) = \frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty} \mathrm{d}s'\, G(x,\gamma_{s'}; s-s')\phi(\gamma_{s'},s')\mathcal{U}(\epsilon; s-s') \tag{A.2}$$

and its complex conjugate

$$\phi^*(x,s) = -\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty} \mathrm{d}s''\, G^*(x,\gamma_{s''}; s-s'')\phi^*(\gamma_{s''},s'')\mathcal{U}(\epsilon; s-s''), \tag{A.3}$$

we get by direct differentiation

$$\begin{aligned}
&\frac{\partial}{\partial s}\Bigg[\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\frac{-\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')\\
&\qquad\times\mathcal{U}(\epsilon;s-s')G(x,\gamma_{s'};s-s')\mathcal{U}(\epsilon;s-s'')G^*(x,\gamma_{s''};s-s'')\Bigg]\\
&\quad=\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\frac{-\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')\\
&\qquad\times\partial_s[G(x,\gamma_{s'};s-s')G^*(x,\gamma_{s''};s-s'')]\mathcal{U}(\epsilon;s-s')\mathcal{U}(\epsilon;s-s'')\\
&\qquad+[\partial_s\mathcal{U}(\epsilon;s-s')\mathcal{U}(\epsilon;s-s'')+\mathcal{U}(\epsilon;s-s')\partial_s\mathcal{U}(\epsilon;s-s'')]\\
&\qquad\times G(x,\gamma_{s'};s-s')G^*(x,\gamma_{s''};s-s'').
\end{aligned} \tag{A.4}$$

Focusing on the first term above, we note that, as G is a homogeneous solution of Schrödinger's equation, we can apply our lemma to that term, which therefore reads

$$\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\frac{-\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')$$
$$\times\,\partial_\mu\bigg[\frac{\mathrm{i}}{2}(D^\mu G(x,\gamma_{s'};s-s')G^*(x,\gamma_{s''};s-s'')$$
$$-(D^\mu G(x,\gamma_{s''};s-s''))^*G(x,\gamma_{s'};s-s'))\bigg]$$
$$\times\,\mathcal{U}(\epsilon;s-s')\mathcal{U}(\epsilon;s-s''). \tag{A.5}$$

Integrating (A.4) with respect to s, the left-hand side vanishes (we can safely assume that it goes to zero for all x, s', s'' as $|s|\to\infty$) and the derivative ∂_μ can be pulled out of the triple integral in the first term. The reader can verify that this triple integral is just $\partial_\mu j^\mu$, with j given by (43) and ϕ, ϕ^* are specified using (A.2) and (A.3). The current j is therefore conserved, if the s integral over the second term in (A.4) vanishes in the limit $\epsilon\to 0$.

Let us then show that, *in the distributional sense* (to be clarified soon), this is indeed the case. Integrating the second term with respect to s and using $\partial_s\mathcal{U}(\epsilon;s-s') = \delta(s-s'-\epsilon)+\delta(s'-s-\epsilon)$, that term reads

$$\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\frac{-\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')$$
$$\times\,\mathcal{U}(\epsilon;s'-\epsilon-s'')G(x,\gamma_{s'};-\epsilon)G^*(x,\gamma_{s''};s'-\epsilon-s'')$$
$$+\mathcal{U}(\epsilon;s'+\epsilon-s'')G(x,\gamma_{s'};+\epsilon)G^*(x,\gamma_{s''};s'+\epsilon-s'')$$
$$+\mathcal{U}(\epsilon;s''-\epsilon-s')G(x,\gamma_{s'};s''-\epsilon-s')G^*(x,\gamma_{s''};-\epsilon)$$
$$+\mathcal{U}(\epsilon;s''+\epsilon-s')G(x,\gamma_{s'};s''+\epsilon-s')G^*(x,\gamma_{s''};+\epsilon). \tag{A.6}$$

Using (A.2) and (A.3), this now reads

$$\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\phi^*(x,s'-\epsilon)G(x,\gamma_{s'};-\epsilon)$$
$$+\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\phi(\gamma_{s'},s')\phi^*(x,s'+\epsilon)G(x,\gamma_{s'};\epsilon)$$
$$-\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')\phi(x,s''-\epsilon)G^*(x,\gamma_{s''};-\epsilon)$$
$$-\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s''\phi^*(\gamma_{s''},s'')\phi(x,s''+\epsilon)G^*(x,\gamma_{s''};+\epsilon). \tag{A.7}$$

By the initial condition (29), $G(x,x';-\epsilon)$,$G(x,x';\epsilon)$, $G^*(x,x';-\epsilon)$ and $G^*(x,x';+\epsilon)$, all converge to the common distribution $\delta^{(4)}(x-x')$. Changing the dummy integration variables above to s and using $\delta^{(4)}(x-\gamma_s)f(x)\sim f(\gamma_s)\delta^{(4)}(x-\gamma_s)$ in the distributional sense, (A.7) now reads

$$\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s\,\delta^{(4)}(x-\gamma_s)(\phi(\gamma_s,s)[\phi^*(\gamma_s,s-\epsilon)+\phi^*(\gamma_s,s+\epsilon)]$$
$$-\phi^*(\gamma_s,s)[\phi(\gamma_s,s-\epsilon)+\phi(\gamma_s,s+\epsilon)]). \tag{A.8}$$

Taylor expanding in ϵ, we get $[\phi(\gamma_s,s-\epsilon)+\phi(\gamma_s,s+\epsilon)] = 2\phi(\gamma_s,s)+R$, where $R=\frac{\partial^2\phi(\gamma_s,s)}{\partial s^2}\epsilon^2$, namely, no linear term in ϵ. Similarly, $[\phi^*(\gamma_s,s-\epsilon)+\phi^*(\gamma_s,s+\epsilon)] = 2\phi^*(\gamma_s,s)+R^*$. Combining these results, we obtain

$$\frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s\,\delta^{(4)}(x-\gamma_s)[\phi(\gamma_s,s)R^*(s)-\phi^*(\gamma_s,s)R(s)]. \tag{A.9}$$

Recall now from section 3.3 that as $\mathcal{N}$ scales like ϵ^{-1}, to get a non-vanishing current, ϕ must also scale as ϵ^{-1} in the limit $\epsilon \to 0$, affecting R. Still, this implies that R is only $O(\epsilon)$—enough to render the above distribution $O(\epsilon)$, establishing our claim that the regularized current is conserved in the distributional sense. Note that the conservation of all the currents in our theory is only meaningful in that sense, namely, we integrate a conserved current j, with $\partial \cdot j = 0$, over a region in space and apply the Stokes theorem to obtain a conserved quantity. What we have just shown is that the second term in (A.4), when integrated over space, vanishes in the limit $\epsilon \to 0$. A conserved quantity can therefore be obtained from the first divergence term.

Appendix B. The classical limit of the central ECD system

The classical limit relies on the 'classical origin' of the proper time Schrödinger equation (28). Starting with a classical Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\dot{x} \cdot \dot{x} + qA(x) \cdot \dot{x}, \tag{B.1}$$

and with a corresponding variational problem for the action functional,

$$I[x] = \int_0^s \mathrm{d}\sigma \, \mathcal{L}(x(\sigma), \dot{x}(\sigma)), \tag{B.2}$$

we get the corresponding Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \equiv qF_{\mu\nu}\dot{x}^\nu - \ddot{x}_\mu = 0, \qquad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{B.3}$$

These are the classical EOM of a charge in an external field F, provided that we define $\mathrm{d}\tau/\mathrm{d}s \equiv m$ the mass of the particle, with $\mathrm{d}\tau = \sqrt{\mathrm{d}x^2}$ the proper time.

The advantage of excluding the mass from the EOM is that now the four components of the velocity are not constrained by a mass-shell condition $\dot{x}^2 = m^2$. Their independence allows us to adopt all the mathematical machinery of the non-relativistic Hamilton–Jacobi theory, with s playing the role of time, and inner products taken in Minkowski's space. In particular, a Hamiltonian can be constructed

$$\mathcal{H}(p, x) := p \cdot \dot{x} - \mathcal{L} = \tfrac{1}{2}(p - qA)^2, \tag{B.4}$$

with

$$p_\mu := \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \dot{x}_\mu + qA_\mu, \tag{B.5}$$

and a corresponding Hamilton–Jacobi equation for the action $I(x, x'; s)$ in (B.2), regarded as a function of the initial (x') and final (x) coordinates of the classical (extremal) path and the interval s:

$$\partial_s I = -\mathcal{H}(\partial_\mu I, x_\mu). \tag{B.6}$$

Note that upon plugging (B.5) into (B.4), we get

$$\mathcal{H} = \tfrac{1}{2}\dot{x}^2, \tag{B.7}$$

which is a constant-of-motion equal to $m^2/2$, as $\mathcal{H}$ is s-independent (or directly by the Poisson brackets formalism, or as shown in section 2). As in the non-relativistic case, classical paths originating from x' at $s = 0$ satisfy

$$\dot{p}^\mu = \frac{\partial I}{\partial x_\mu}(x, x'; s). \tag{B.8}$$

Pursuing the analogy with the non-relativistic case, we can now quantize the relativistic Hamiltonian (B.4). This amounts (up to the problem of operator ordering) to the substitution

$$p_\mu \mapsto \hat{p}_\mu, \qquad x_\mu \mapsto \hat{x}_\mu, \qquad [\hat{x}_\mu, \hat{p}_\nu] = \mathrm{i} g_{\mu\nu},$$

where $\hat{p}_\mu$ and $\hat{x}_\mu$ are operators acting on a Hilbert space spanned by their (normalized) eigenvectors: $\hat{p}_\mu|p\rangle = p_\mu|p\rangle, \hat{x}_\mu|x\rangle = x_\mu|x\rangle$. It follows that $|x\rangle$ and $|p\rangle$ satisfy

$$\langle x|x'\rangle = \delta^{(4)}(x-x'), \qquad \langle p|p'\rangle = \delta^{(4)}(p-p'), \qquad \langle x|p\rangle = \frac{1}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}p\cdot x}. \tag{B.9}$$

Based on the formal substitution $t \mapsto s$, the s-evolution of the state vector $|\phi(s)\rangle$ now reads

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}|\phi\rangle = \hat{\mathcal{H}}|\phi\rangle, \tag{B.10}$$

and upon projecting equations (B.10) on $\langle x|$, a Schrödinger-like equation for $\phi(x,s) := \langle x|\phi(s)\rangle$ is obtained:

$$\mathrm{i}\partial_s\phi = \tfrac{1}{2}(-\mathrm{i}\partial_\mu - qA_\mu)(-\mathrm{i}\partial^\mu - qA^\mu)\phi. \tag{B.11}$$

The next ingredient needed to establish the classical limit of the system (21) and (26) is the semiclassical expression for the propagator, G, entering (26), defined by (28) and (29). By utilizing relations (B.9), the construction of a path-integral representation to G can be carried out in full analogy to the non-relativistic case, resulting in

$$G(x,x';s) = \int \mathcal{D}[\beta]\,\mathrm{e}^{\mathrm{i}I_\beta(x,x';s)/\hbar}, \tag{B.12}$$

where I_β is the action (B.2) of the path β. The paths which enter the integral are only restricted by the boundary conditions $\beta(0) = x'$ and $\beta(s) = x$ and are not constrained to lie on a single mass shell—whether positive or negative.

Continuing as in the non-relativistic path integral, we can construct the so-called 'semi-classical' propagator by expanding the action around its stationary points, which are the classical paths y (including tachyonic paths) that satisfy the boundary conditions

$$G_{\mathrm{sc}}(x,x';s) = \frac{\mathrm{i}\,\mathrm{sign}(s)}{(2\pi\hbar)^2}\sum_\beta \mathcal{F}_\beta(x,x';s)\,\mathrm{e}^{\mathrm{i}I_\beta(x,x';s)/\hbar}, \tag{B.13}$$

where $\mathcal{F}_\beta$—the so-called Van–Vleck determinant—is a purely classical quantity, given by the determinant

$$\mathcal{F}_\beta(x,x';s) = \left|-\partial_{x_\mu}\partial_{x'_\nu} I_\beta(x,x';s)\right|^{1/2}. \tag{B.14}$$

The two classic results of path integrals which we utilize are that, for quadratic Lagrangians, $\mathcal{F}$ does not depend on x and x', and that the semiclassical propagator for such Lagrangians is exact. Of immediate importance is the Van Vleck determinant for the case of a linear potential—a constant field F:

$$\bar{\mathcal{F}}(s) = s^{-2}|\det g(qFs)|^{\frac{1}{2}}, \qquad \text{with} \quad g(y) = \frac{y^2}{2-2\cosh y} = 1 + O(y^2). \tag{B.15}$$

Equipped with the necessary links between classical mechanics and propagator theory, we can move on to the task of establishing that, under certain conditions, solutions to the ECD system (21) and (26) have for their γ part a classical path, i.e. a solution of the classical EOM (B.3). We assume that the exact propagator G entering (26) is faithfully represented by the corresponding semiclassical propagator (B.13).[5] Under this assumption, (26) now reads

$$\phi(x,s) = \frac{\mathrm{i}}{\mathcal{N}}\int_{-\infty}^{\infty}\mathrm{d}s'\sum_\beta \frac{\mathrm{i}\,\mathrm{sign}(s-s')}{(2\pi\hbar)^2}\mathcal{F}_\beta(x,\gamma_{s'};s-s')\,\mathrm{e}^{\mathrm{i}I_\beta(x,\gamma_{s'};s-s')/\hbar}\,\phi(\gamma_{s'},s')\mathcal{U}(\epsilon;s-s'), \tag{B.16}$$

[5] At least in the non-relativistic theory, the semiclassical propagator is *exact* in all exactly solvable cases. The name 'semiclassical' is therefore misleading.

with β standing for a one-parameter family of classical paths in the external field F, parameterized by s', connecting $\gamma_{s'}$ with x.

We next plug in the ansatz

$$\phi(\gamma_s, s) = C \exp \mathrm{i} I_\gamma\,[\gamma(s), \gamma(0); s]\,/\bar{h}, \tag{B.17}$$

with $C \in \mathbb{C}$ on the right-hand side of (B.16) (generalizing (34) of section 3.2.1), carry the s' integral, substitute $x = \gamma_s$ and check for self-consistency, that is, that the left-hand side of (B.16) indeed equals (B.17). We further assume that the contributions of the *indirect paths* β's to the sum in (B.16) can be ignored. By indirect paths we mean classical paths connecting $\gamma_{s'}$ with γ_s not via γ itself, e.g. a classical trajectory origination from $\gamma_{s'}$, bouncing off a remote potential and back to γ_s. This delicate assumption is justified below. Using (B.2) and (B.1), the right-hand side of (B.16) reads

$$\frac{\mathrm{i}C}{\mathcal{N}} \exp \mathrm{i} I_\gamma\,[\gamma(s), \gamma(0); s]\,/\bar{h} \int_{-\infty}^{\infty} \mathrm{d}s'\, \frac{\mathrm{i}\,\mathrm{sign}(s-s')}{(2\pi\bar{h})^2} \mathcal{F}_\gamma(\gamma_s, \gamma_{s'}; s-s')\mathcal{U}(\epsilon; s-s'),$$

where the subscript γ under $\mathcal{F}$ and I indicates that a preferred—direct—path, i.e. γ, connecting $\gamma_{s'}$ with $x = \gamma_s$ via γ, has been singled out. As $\mathcal{F}_\gamma(\gamma_s, \gamma_{s'}; s-s')$ has the same $(s-s')^{-2}$ divergence as the the free propagator (33), the above integral has the same ϵ^{-1} divergence as in the free case (36). Equation (26) is therefore satisfied as long as finite, namely ϵ-independent, corrections to the integral are negligible. The dominant contribution comes from s' for which $\mathcal{F}_\gamma(\gamma_s, \gamma_{s'}; s-s') \sim \bar{\mathcal{F}}(s-s')$, leading to a correction which is $O(q^2F^2)$, which is consistent with the omission of the self-force.

We move now to the second ECD equation—equation (21)—written in the alternative form (40). We can use (B.16) to express $\partial_\mu\phi(\gamma_s, s)$ in (40) by 'pushing' the derivative into the integral

$$\begin{aligned}\partial_\mu\phi(\gamma_s, s) = \frac{\mathrm{i}C}{\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}s' \big[&\mathrm{i}\partial_\mu I_\gamma(x, \gamma_{s'}; s-s')|_{x=\gamma_s} \mathcal{F}_\gamma(\gamma_s, \gamma_{s'}; s-s')/\bar{h} \\ &+ \partial_\mu \mathcal{F}_\gamma(x, \gamma_{s'}; s-s')|_{x=\gamma_s} \big] \frac{\mathrm{i}\,\mathrm{sign}(s-s')}{(2\pi\bar{h})^2} \mathrm{e}^{\mathrm{i}I_\gamma(\gamma_s, \gamma_{s'}; s-s')/\bar{h}} \phi(\gamma_{s'}, s')\mathcal{U}(\epsilon; s-s').\end{aligned} \tag{B.18}$$

The classical limit follows when the second term in the square brackets above is negligible compared with the first one which, by (B.8), is just $\mathcal{F}/\bar{h}$ times the momentum, p_μ, conjugate to γ_μ, of a trajectory originating from $\gamma_{s'}$ and 'proper time' s', ending at γ_s and proper time s. $\partial_\mu I_\gamma(x, \gamma_{s'}; s-s')|_{x=\gamma_s}$ is therefore independent of s' and can be pulled out of the integral, resulting in

$$\partial_\mu\phi(\gamma_s, s) = \mathrm{i}\bar{h}^{-1} p_\mu(s)\phi(\gamma_s, s) \;\Rightarrow\; \partial_\mu\phi(\gamma_s, s)\phi^*(\gamma_s, s) = \mathrm{i}\bar{h}^{-1} p_\mu(s)|\phi(\gamma_s, s)|^2, \tag{B.19}$$

and (40) is satisfied independently of the value of $\bar{h}$. In the rest-frame of the charge $(p = (m, 0, 0, 0))$, the classical limit appears as the limit $m/\bar{h} \to \infty$ or $\lambda_e \to 0$.

As λ_e has dimension 1, the scale and gauge invariant extension of the above statement is $\lambda_e/\lambda_F \ll 1$, with λ_F being some characteristic measure of the scale of variation of F, with dimension 1. The relevance of λ_F to the classical limit can directly be seen by analyzing the omitted indirect-path contributions to ϕ in (B.16) and (B.18). Rather than exploring the general theory, we shall focus on one simple exactly solvable example, that of a scalar, time-independent 'delta function potential' located at the origin of three space. A typical propagator for such a Hamiltonian takes the form

$$G = G_{\mathrm{f}} + \mathrm{sign}(s-s') \frac{\exp \mathrm{i} \frac{(x^0 - x'^0)^2 - (|\boldsymbol{x}| + |\boldsymbol{x}'|)^2}{2\bar{h}(s-s')}}{(2\pi\bar{h})^2 |\boldsymbol{x}||\boldsymbol{x}'|(s-s')}, \tag{B.20}$$

with G_{f} the free propagator (33). It satisfies a free Schrödinger's equation in a modified Hilbert space defined by the boundary condition

$$\nabla(Gr)|_{r=0} \equiv 0. \tag{B.21}$$

The phase of the second term is just the action of the indirect classical path originating at s' from $(x'^0, \boldsymbol{x}')$, elastically bouncing off the origin and reaching $(x^0, \boldsymbol{x})$ at s. The second term is therefore a prototype of an indirect contribution. Its contribution to (B.16) and (B.18) is strongly suppressed for $\lambda_e/r \ll 1$ as the phase of the s' integral does not have a stationary point. This implies that ECD charges traveling far from the origin compared to their Compton length would experience only marginal *local* deviations from straight classical paths. However, very small deviations can accumulate to an overall global effect which can be amplified by suitable experimental settings, such as a large distance to the screen in a scattering experiment. Such indirect contributions are therefore at the heart of ECD's 'remote sensing' mechanism, which is a mandatory feature in any hidden variables model.

Finally, in a spin-$\frac{1}{2}$ variant of ECD, the classical limit is unaltered. Paraphrasing the result of Schwinger in [4], we see that the propagator in a constant field F gets multiplied by the spinor propagator $\mathrm{e}^{-\mathrm{i}\mathcal{H}_\sigma(s-s')}$, with $\mathcal{H}_\sigma = \frac{q}{2}\sigma_{\mu\nu}F^{\mu\nu}(x)$. It follows that if the ansatz (B.17) is multiplied by $\mathrm{e}^{-\mathrm{i}\mathcal{H}_\sigma s}\chi_0$ for any constant bispinor χ_0 (constant precession compensating for $\mathrm{e}^{-\mathrm{i}\mathcal{H}_\sigma(s-s')}$), then classical trajectories in a constant field are still exact ECD solutions. Moreover, as in the scalar case, if $F(x)$ varies slowly on the charge's Compton length scale, then the bispinor $\phi(\gamma_s, s)$ precesses along the local $F(\gamma_s)$ and does not affect the dynamics of γ. However, when $F(x)$ varies rapidly on the Compton length scale (in Compton scattering or near the nucleus of an atom), there is a nontrivial coupling between γ and ϕ with no classical counterpart (even when $F(x)$ varies slowly on the Compton length scale, as in a Stern–Gerlach experiment, that small coupling can still be amplified by placing the detector far from the apparatus).

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